JOURNAL OF APPROXIMATION THEORY 70, 273–283 (1992)

Multivariate Bernstein and Markov Inequalities

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Communicated by Paul Nevai

Received March 11, 1991; revised June 22, 1991

For a polynomial P_n of total degree n and a bounded convex set S it will be shown that for $0 < p \leqslant \infty$

$$\left\| \frac{\partial}{\partial \xi} P_n \right\|_{L_p(S)} \leqslant Cn^2 \, \| P_n \|_{L_p(S)}$$

with C independent of n and of $P_n \in \Pi_n$. The Bernstein inequality

$$\left\| \sqrt{1-x^2} \frac{d}{dx} P_n(x) \right\|_{L_p[-1,1]} \leqslant Cn \, \| P_n \|_{L_p[-1,1]}$$

will also be generalized and that generalization will be the crucial result. Theorems for higher and mixed derivatives will be achieved.

1. INTRODUCTION

For a measurable set S the L_p norm or quasi-norm is given as usual by

$$\| f \|_{L_p(S)} = \begin{cases} \left(\int_S | f |^p \, d\mu \right)^{1/p}, & 0 < p < \infty \\ \operatorname*{ess\,sup}_{x \in S} | f(x) |, & p = \infty. \end{cases} \tag{1.1}$$

The Bernstein inequality for trigonometric polynomials of degree n, T_n, is given by

$$\| T_n' \|_{L_p[-\pi,\pi]} \leqslant n \, \| T_n \|_{L_p[-\pi,\pi]}, \qquad 0 < p \leqslant \infty \tag{1.2}$$

and was proved by S. N. Bernstein for $p = \infty$, by A. Zygmund for $1 \leqslant p < \infty$, and, more recently, by Arestov [1] for $0 < p < 1$.

* Supported by NSERC Grant A-4816 of Canada.

0021-9045/92 $5.00

For algebraic polynomials of degree n, P_n, one can show

$$\| P'_n \|_{L_p[-1,1]} \leqslant Cn^2 \| P_n \|_{L_p[-1,1]}, \qquad 0 < p \leqslant \infty \tag{1.3}$$

and

$$\| \varphi P'_n \|_{L_p[-1,1]} \leqslant Cn \| P_n \|_{L_p[-1,1]}, \qquad 0 < p \leqslant \infty, \varphi(x)^2 = 1 - x^2 \tag{1.4}$$

which one may call the Markov and the Bernstein inequality, respectively. In fact, (1.3) was proven by Marlov for $p = \infty$ and by Hille, Szegő, and Tamarkin [6] for $1 \leqslant p < \infty$, and it is stated for $0 < p \leqslant 1$ in [7]. The inequality (1.4), which is a derivate of (1.2), is a copy of (1.2) for $p = \infty$ and is given explicitly for other p when one substitutes $W = W_n = 1$ in [9, Theorem 5] (see also [3, Theorem 8 4]). We remark that (1.3) can be derived from (1.4) for $0 < p < 1$ (see S[illegible]n 4) and hence, we do not really rely on the unpublished proof of [[illegible]].

For multivariate trigonometric polynomials, degree is a direction dependent property, and works on the subject reflect that fact and use derivatives in the axes' directions. For algebraic polynomials, one may consider polynomials of total degree, a concept which is independent of the directions of the axes. Hence, multivariate generalizations of (1.3) and (1.4) may take a form which does not follow from the multivariate trigonometric case in the way that (1.3) and (1.4) follow from (1.2). In fact, one uses (1.3) and (1.4) to obtain their multivariate analogue.

The result achieved in this paper attempts to obtain easy to use estimates rather than the most general ones. For a point $v \in S$, the directional distance from the boundary, $\tilde{d}(v, \xi)$, is given by

$$\tilde{d}(v, \xi) \equiv \sup_{\substack{0 < \lambda \\ v + \lambda\xi \in S}} d(v, v + \lambda\xi) \sup_{\substack{\lambda < 0 \\ v + \lambda\xi \in S}} d(v, v + \lambda\xi). \tag{1.5}$$

The main result of the paper is

$$\left\| \tilde{d}(\cdot, \xi)^{r/2} \left(\frac{\partial}{\partial \xi} \right)^r P_n(\cdot) \right\|_{L_p(S)} \leqslant C(r, p)\, n^r \, \| P_n(\cdot) \|_{L_p(S)}. \tag{1.6}$$

We will deduce a Markov type inequality from (1.6) and a discussion of the "main-domain" of P_n.

We found out that a large portion of our result in Section 4 on the Markov inequality was superceded by results of P. Goetgheluck [4, 5]. He treated Lipschitzian compact sets and $1 \leqslant p \leqslant \infty$ instead of compact convex sets and $0 < p \leqslant \infty$ here. I believe that since the result in Section 4 was easily extended to all $p > 0$ and the proof was relatively simple, Section 4 is still a worthwhile contribution. I believe that Goetgheluck's theorem will extend to all $p > 0$ and hope that this will be proved in the near future (maybe by P. Goetgheluck). In fact, the history of results related to Section 4 is very rich. Multivariate Markov inequalities seem to start with Coatmelec [2] and were pursued by Paułucki, Plesniak, and others (see [10, 11]). In particular, one should mention H. Wallin who wrote many articles on the subject. Nevertheless, the results of Section 4 here which are useful with the present simple conditions do not follow completely from these articles.

2. The Multivariate Bernstein Inequality

In this section we will prove the following result which is a Bernstein type inequality.

THEOREM 2.1. *For a bounded convex set $S \subset R^d$, any direction ξ ($|\xi| = 1$ where $|\xi|$ is the Euclidean norm of ξ), integer r, and $0 < p \leqslant \infty$, we have*

$$\left\| \tilde{d}(\cdot, \xi)^{r/2} \left(\frac{\partial}{\partial \xi}\right)^r P_n(\cdot) \right\|_{L_p(S)} \leqslant C(p, r)\, n^r \, \| P_n(\cdot) \|_{L_p(S)}, \tag{2.1}$$

where $C(p, r)$ is independent of S, ξ, n, and P_n.

Remark. If $S^\circ = \varnothing$, the result is trivial for $L_p(S)$. For $f \in C(S)$, one would have an analogue of (2.1) even for $S^\circ = \varnothing$ if ξ is restricted sufficiently (see also Remark 4.3).

Proof. For a vector ξ and a convex set S, we define $S(\xi)$ as the orthogonal projection of S on $R^{d-1}(\xi)$ where $R^{d-1}(\xi) \perp \xi$. We now define for $u \in S(\xi)$,

$$\lambda_1 = \lambda_1(u, \xi) \equiv \operatorname{Inf}(\lambda; u + \lambda\xi \in S)$$

and (2.2)

$$\lambda_2 = \lambda_2(u, \xi) \equiv \operatorname{Sup}(\lambda; u + \lambda\xi \in S).$$

For $0 < p < \infty$, we write

$$\begin{aligned} &\left\| \tilde{d}(\cdot, \xi)^{r/2} \left(\frac{\partial}{\partial \xi}\right)^r P_n(\cdot) \right\|_{L_p(S)} \\ &= \left\{ \int_{S(\xi)} \int_{\lambda_1(u, \xi)}^{\lambda_2(u, \xi)} \left| \tilde{d}(u + \lambda\xi, \xi)^{r/2} \left(\frac{\partial}{\partial \xi}\right)^r P_n(u + \lambda\xi) \right|^p d\lambda\, du \right\}^{1/p} \\ &\equiv \left\{ \int_{S(\xi)} I_n(\xi, u)\, du \right\}^{1/p}. \end{aligned}$$

To estimate $I_n(\xi, u)$, we write

$$I_n(\xi, u) = \int_{\lambda_1}^{\lambda_2} \left| ((\lambda-\lambda_1)(\lambda_2-\lambda))^{r/2} \left(\frac{\partial}{\partial\lambda}\right)^r P_n(u+\lambda\xi)\right|^p d\lambda$$
$$= \frac{\lambda_2-\lambda_1}{2} \int_{-1}^{1} \left| (1-\mu^2)^{r/2} \left(\frac{\partial}{\partial\mu}\right)^r \right.$$
$$\left. \times P_n\left(u+\mu\left(\frac{\lambda_2-\lambda_1}{2}\right)\xi + \frac{\lambda_1+\lambda_2}{2}\xi\right)\right|^p d\mu,$$

where $\mu = 2((\lambda-\lambda_1)/(\lambda_2-\lambda_1))-1$, $1-\mu^2 = (2/(\lambda_2-\lambda_1))^2(\lambda-\lambda_1)(\lambda_2-\lambda)$ and $\partial/\partial\mu = ((\lambda_2-\lambda_1)/2)(\partial/\partial\lambda)$. Since for fixed u and ξ, $P_n(u+\mu((\lambda_2-\lambda_1)/2)\xi + ((\lambda_1+\lambda_2)/2)\xi)$ is a polynomial in μ of degree n or smaller, we use the known result (1.3) to obtain

$$I_n(\xi, \mu) \leqslant \frac{\lambda_2-\lambda_1}{2}$$
$$\times C(p, r)^p n^{rp} \int_{-1}^{1} \left| P_n\left(u+\mu\left(\frac{\lambda_2-\lambda_1}{2}\right)\xi + \frac{\lambda_1+\lambda_2}{2}\xi\right)\right|^p d\mu$$
$$= C(p, r)^p n^{rp} \int_{\lambda_1}^{\lambda_2} |P_n(u+\lambda\xi)|^p d\lambda$$

with $C(p, r)$ depending on p and r but not on n or P_n. Therefore,

$$\left\{\int_{S(\xi)} I_n(\xi, u)\, du\right\}^{1/p} \leqslant C(p, r)\, n^r \left\{\int_{S(\xi)} \int_{\lambda_1}^{\lambda_2} |P_n(u+\lambda\xi)|^p\, d\lambda\, du\right\}^{1/p}$$
$$= C(p, r)\, n^r \, \|P_n\|_{L_p(S)}$$

which completes the proof for $0 < p < \infty$. For $p = \infty$, we have

$$\left\| \tilde{d}(\cdot, \xi)^{r/2} \left(\frac{\partial}{\partial\xi}\right)^r P_n(\cdot)\right\|_{L_\infty(S)}$$
$$= \operatorname*{Sup}_{u\in S(\xi)} \operatorname*{Sup}_{\lambda_1(\xi,u)\leqslant\lambda\leqslant\lambda_2(\xi,u)} \left| ((\lambda-\lambda_1)(\lambda_2-\lambda))^{r/2} \left(\frac{\partial}{\partial\lambda}\right)^r P_n(r+\lambda\xi)\right|$$
$$\leqslant \operatorname*{Sup}_{u\in S(\xi)} \operatorname*{Sup}_{|\mu|\leqslant 1} \left| (1-\mu^2)^{r/2} \left(\frac{\partial}{\partial\mu}\right)^r P_n\left(u+\mu\left(\frac{\lambda_2-\lambda_1}{2}\right)\xi + \frac{\lambda_1+\lambda_2}{2}\xi\right)\right|$$
$$\leqslant C(r)\, n^r \operatorname*{Sup}_{u\in S(\xi)} \operatorname*{Sup}_{|\mu|\leqslant 1} \left| P_n\left(u+\mu\left(\frac{\lambda_2-\lambda_1}{2}\right)\xi + \frac{\lambda_1+\lambda_2}{2}\xi\right)\right|$$
$$\leqslant C(r)\, n^r \, \|P_n(\cdot)\|_{L_\infty(S)}. \quad \blacksquare$$

From the proof of Theorem 2.1, we observe that we actually have:

THEOREM 2.2. *If (instead of assuming that S is convex) we assume that $S \cap \{x + t\xi : t \in R\}$ is convex for all x, then* (2.1) *is still valid for that ξ with $C(p, r)$ of Theorem* 2.1.

3. MAIN DOMAIN

In this section, we will show that we can reduce the domain S a little without changing the order of magnitude of the norm or quasi-norm of P_n. In one dimension, the estimate of this type was given for $0 < p \leqslant \infty$ by

$$\| P_n \|_{L_p[-1,1]} \leqslant C(p, A) \| P_n \|_{L_p[-1 + An^{-2}, 1 - An^{-2}]}, \tag{3.1}$$

where $C(p, A)$ is independent of n and P_n. (See for instance for $0 < p < \infty$ [9, Lemma 3], setting there $\alpha = \beta = \gamma = 0$ and repeating with δ to get A, and for $p \geqslant 1$, see [3, Theorem 8.4.8], setting there $W = 1$, see also [8].) We define S_ε for a set S by

$$S_\varepsilon = \{u \in S : \{v : |u - v| \leqslant \varepsilon\} \subset S\}, \tag{3.2}$$

where $|u|$ denotes the Euclidean norm of u.

It is obvious that if S is convex, so is S_ε and that $S_\varepsilon \subset S$. We are now ready to state and prove that main reslt of this section.

THEOREM 3.1. *For any bounded convex set S, $A > 0$, $0 < p \leqslant \infty$, and $n \geqslant n_0(A, S)$,*

$$\| P_n \|_{L_p(S)} \leqslant C(p, A, S) \| P_n \|_{L_p(S_{An^{-2}})}, \qquad P_n \in \Pi_n \tag{3.3}$$

where Π_n is the collection of polynomials of total degree n.

We note that in Theorem 3.1 the constant $C(p, A, S)$ in (3.3) depends on S while in Theorem 2.1 the constant in (2.1) does not. This fact is inherent as in Theorem 2.1 the factor $\tilde{d}(\cdot, \xi)$ compensates for the geometric structure of S.

We will need the following Lemma.

LEMMA 3.2. *Suppose V is a bounded convex set satisfying for the direction ξ*

$$\operatorname*{Inf}_{u \in V} \ \operatorname{Sup} \{ |\lambda_2 - \lambda_1| : u + \lambda_i \xi \in V, i = 1, 2\} \geqslant r/2 \tag{3.4}$$

and define $V(\xi, Bn^{-2})$ *by*

$$V(\xi, Bn^{-2}) = \{u \in V : (u - Bn^{-2}\xi, u + Bn^{-2}\xi) \subset V\}. \tag{3.5}$$

Then for $0 < p \leqslant \infty$ *and* $n \geqslant n_0(d, B)$,

$$\| P_n \|_{L_p(V)} \leqslant C(r, B) \| P_n \|_{L_p(V(\xi, Bn^{-2}))}, \tag{3.6}$$

where $C(r, B)$ *depends neither on the shape of* V *nor on* n.

Proof of Lemma 3.2. We define $V(\xi)$ as we defined $S(\xi)$ in the proof of Theorem 2.1. That is, $V(\xi)$ is the orthogonal projection of V on $R^{d-1}(\xi)$ where $R^{d-1}(\xi) \perp \xi$. For $u \in V(\xi)$, $\lambda_i(u, \xi)$ $(i = 1, 2)$ is given by

$$\lambda_1(u, \xi) = \text{Inf}\{\lambda : u + \lambda\xi \in V\}$$

and

$$\lambda_2(u, \xi) = \text{Sup}\{\lambda : u + \lambda\xi \in V\}.$$

We observe that (3.4) implies

$$\lambda_2(u, \xi) - \lambda_1(u, \xi) \geqslant r/2.$$

For $0 < p < \infty$, we write

$$\| P_n(\cdot) \|_{L_p(V)} \leqslant \left\{ \int_{V(\xi)} \int_{\lambda_1(u, \xi)}^{\lambda_2(u, \xi)} |P_n(u + \lambda\xi)|^p \, d\lambda \, du \right\}^{1/p}$$

$$\equiv \left\{ \int_{V(\xi)} I_n(\xi, u) \, du \right\}^{1/p}.$$

We now estimate $I_n(\xi, u)$ by

$$I_n(\xi, u) = \frac{\lambda_2 - \lambda_1}{2} \int_{-1}^{1} \left| P_n\left(u + \mu\left(\frac{\lambda_2 - \lambda_1}{2}\right)\xi + \frac{\lambda_1 + \lambda_2}{2}\xi\right)\right|^p d\mu$$

$$\leqslant \frac{\lambda_2 - \lambda_1}{2} C(B_1, p)^p$$

$$\times \int_{-1 + B_1 n^{-2}}^{1 - B_1 n^{-2}} \left| P_n\left(u + \mu\left(\frac{\lambda_2 - \lambda_1}{2}\right)\xi + \frac{\lambda_1 + \lambda_2}{2}\right)\right|^p d\mu$$

$$\leqslant C(B_1, p)^p \int_{\lambda_1 + Bn^{-2}}^{\lambda_2 - Bn^{-2}} |P_n(u + \lambda\xi)|^p \, d\lambda$$

with $B_1((\lambda_2-\lambda_1)/2)\geqslant B$ or $B_1(r/4)\geqslant B$. The proof for L_∞ follows standard changes when we observe that P_n is continuous. ∎

Proof of Theorem 3.1. In order to prove Theorem 3.1, it is sufficient to construct a finite number of V_i satisfying:

(a) inequality (3.4) is valid for all V_i with respect to some ξ_i,

(b) $\bigcup_{i=1}^{l} V_i = S$, and

(c) $V_i(\xi_i, Bn^{-2})\subset S_{An^{-2}}$ for some B.

If the above are satisfied, we have

$$\begin{aligned}\|P_n\|_{L_p(S)} &\leqslant C\sum_{i=1}^{l}\|P_n\|_{L_p(V_i)}\\ &\leqslant C\sum_{i=1}^{l} C(B,p)\|P_n\|_{L_p(V_i(\xi_i,Bn^{-2}))}\\ &\leqslant l\cdot C\cdot C(B,p)\|P_n\|_{L_p(S_{An^{-2}})}\end{aligned}$$

(where $C=1$ for $1\leqslant p\leqslant\infty$) and hence, the construction of V_i will yield the proof of our theorem.

We define V_i first. If $\|P_n\|_{L_p(S)}=0$, the inequality (3.3) is trivial. For $0<p\leqslant\infty$, a convex set S and $\|P_n\|_{L_p(S)}\neq 0$, S contains a ball U of d dimensions with radius $r>0$ and center u_0. We define

$$V_i=\{S\cap\{x: x=\tfrac{1}{3}(u-u_0)+u_0+\lambda\xi,\ u\in U,\ \lambda\in R\}\}$$

which is an intersection of S with a cylinder (d dimensional) with center u_0, direction ξ, and radius $r/3$. As the diameter of S is finite, say L, a finite number of those V_ξ will cover S. In fact, a bound of the number can be given in terms of L, r, and the dimension d.

It only remains to prove that for some B depending on A, r, and L, we have

$$V_i(\xi_i, Bn^{-2})\subset S_{An^{-2}}.$$

This follows from the fact that

$$S\supset \text{conv hull}\{V_i\cup U\}=T$$

and hence

$$S_{An^{-2}}\supset T_{An^{-2}}.$$

We now observe that for $T_{An^{-2}}$ to contain $V(\xi_i, Bn^{-2})$, we have only to choose B sufficiently large, where B depends on the ratio L/r. ▮

Remark 3.3. We note that for $p=\infty$, it is possible for a convex set S and a polynomial P_n to satisfy $\|P_n(\cdot)\|_{C(S)} \neq 0$ while $S^\circ = \emptyset$. Of course $S_{An^{-2}} \subset S^\circ$ and hence, (3.3) will not always be valid with the C norm if $S^\circ = \emptyset$. However, in such a case, if S contains more than one point, $S \subset u + T(R^{d-j})$ where R^{d-j} is the $d-j$ $(d-j>0)$ dimensional Euclidean space, and T is a regular linear transformation. In this case, S has an interior point in the R^{d-j} sense. The polynomial P_n is at most of nth degree on the map of R^{d-j} and Theorem 3.1 can be restated in $u+T(R^{d-j})$ with the C norm (or in the L_∞ norm) restricted to $u+T(R^{d-j})$.

4. The Markov Inequality

In this section, we will give a multivariate version of the Markov inequality. (The case $p=\infty$ will be discussed further in Remark 4.3.) See also [4, 5, 10, 11].

Theorem 4.1. *For a bounded convex set S, $0<p\leqslant\infty$, a direction ξ, and a polynomial P_n of total degree n, we have*

$$\left\| \frac{\partial}{\partial \xi} P_n(\cdot) \right\|_{L_p(S)} \leqslant C(p, S)\, n^2 \, \|P_n(\cdot)\|_{L_p(S)}, \tag{4.1}$$

where $C(p, S)$ depends only on p and S.

We note that, in fact, $S^\circ \neq \emptyset$ as $S^\circ = \emptyset$ implies for $0<p\leqslant\infty$ that both expressions in (4.1) are equal to 0. Theorem 4.1 implies the following corollary.

Corollary 4.2. *Under the assumptions of Theorem* 4.1, *we have for directions $\xi_1, ..., \xi_k$, the inequality*

$$\left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} P_n(\cdot) \right\|_{L_p(S)} \leqslant C(p, k, S)\, n^{2k} \, \|P_n(\cdot)\|_{L_p(S)}. \tag{4.2}$$

Proof of Corollary 4.2. Repeating (4.1) k times, we have (4.2). ▮

Remark 4.3. For $p=\infty$, $\|P_n(\cdot)\|_{L_\infty(S)}=0$ if $S^\circ=\emptyset$ but $\|P_n(\cdot)\|_{C(S)}$ may be different from 0 even if $S^\circ=\emptyset$. In this case, there is a j such that

$u_0 + T(R^{d-j}) \supset S$. The smallest number $d-j$ for any u_0 is unique. If $d-j \geqslant 1$, we translate the space by u_0 and in the topology of $T(R^{d-j})$, there is an interior point. In this case, Theorem 4.1 is still valid for $\xi \in R^{R-j}$. For $L_p(S)$ the result is valid trivially in case $S^\circ = \varnothing$.

Proof of Theorem 4.1. For $0 < p \leqslant \infty$, we have

$$\left\| \frac{\partial}{\partial \xi} P_n(\cdot) \right\|_{L_p(\bar{S})} = \left\| \frac{\partial}{\partial \xi} P_n(\cdot) \right\|_{L_p(S^\circ)} \tag{4.3}$$

for a convex set S with an interior point and hence we may assume that S is closed (or open). To prove our theorem, it is sufficient to show that there exists a finite collection of sets V_i satisfying the following properties:

(a) To each V_i there is a direction ξ_i such that

$$V_i = \bigcup_{\lambda \in R} (V_i + \lambda \xi_i) \cap S. \tag{4.4}$$

(b) For every point $x \in V_i$, $\operatorname{Sup}_{x+\lambda_j \xi_i \in V_i} |\lambda_2 - \lambda_1| \geqslant r$ with respect to its direction ξ_i.

(c) For every point $z \in S$, $\{x : |x-z| < r\} \cap S$ is covered by d of the sets V_i, say V_{i_j}, $1 \leqslant j \leqslant d$, such that the directions ξ_{i_j} are independent. This implies that there exists a finite cover U_l such that each set U_l is covered by d of the V_i with independent ξ_i satisfying (a), (b) and instead of (c) for every $x_0 \in S$, $\{x : |x - x_0| < r/2\} \cap S$ is in some U_l.

Assuming that we can construct the V_i and U_l as described, it is sufficient to estimate $\|(\partial/\partial\zeta) P_n(\cdot)\|_{L_p(U_l)}$ by

$$\begin{aligned} \left\| \frac{\partial}{\partial \zeta} P_n(\cdot) \right\|_{L_p(U_l)} &\leqslant C \operatorname*{Sup}_{1 \leqslant j \leqslant d} \left\| \frac{\partial}{\partial \xi_{i_j}} P_n(\cdot) \right\|_{L_p(U_l)} \\ &\leqslant C \operatorname*{Sup}_{1 \leqslant j \leqslant d} \left\| \frac{\partial}{\partial \xi_{i_j}} P_n(\cdot) \right\|_{L_p(V_{ij})} \end{aligned} \tag{4.5}$$

With no loss of generality, we call the direction ξ (instead of ξ_{i_j}) and the set V instead of V_{i_j}. We observe that even if the constants depend on these quantities, i.e., on ξ on V, it would not make a difference as there are only a finite number of them to be considered. Using the fact that $(\partial/\partial\xi) P_n$ is of total degree smaller than n, we utilize Lemma 3.2 to obtain

$$\left\| \frac{\partial}{\partial \xi} P_n(\cdot) \right\|_{L_p(V)} \leqslant C(r, B) \left\| \frac{\partial}{\partial \xi} P_n(\cdot) \right\|_{L_p(V(\xi, Bn^{-2}))}, \tag{4.6}$$

where $V(\xi, Bn^{-2})$ is given by (3.5). We now write for some fixed B,

$$\begin{aligned}\left\| \frac{\partial}{\partial\xi} P_n(\cdot) \right\|_{L_p(V(\xi, Bn^{-2}))} &\leqslant Cn \left\| \tilde{d}(\cdot, \xi)^{1/2} \frac{\partial}{\partial\xi} P_n(\cdot) \right\|_{L_p(V(\xi, Bn^{-2}))} \\ &\leqslant Cn \left\| \tilde{d}(\cdot, \xi)^{1/2} \frac{\partial}{\partial\xi} P_n(\cdot) \right\|_{L_p(V)} \\ &\leqslant Cn \left\| \tilde{d}(\cdot, \xi)^{1/2} \frac{\partial}{\partial\xi} P_n(\cdot) \right\|_{L_p(S)} \leqslant C_1 n^2 \, \|P_n(\cdot)\|_{L_p(S)}.\end{aligned}$$

Therefore, we only have to show that the construction described above is possible.

A convex set S with an interior point x_0 has a ball of radius $3r$, $\{x: |x-x_0| < 3r\}$ inside S. Take any direction ξ and observe the sets (cylinders) $A_{\pm} = \{\lambda\xi + x: \lambda \in R_{\pm}, |x-x_0| < r\}$. We will now show that there are d sets V_{i_j}, $1 \leqslant j \leqslant d$, with d independent directions that cover $A_+ \cap S$ (or $A_- \cap S$) and this will be sufficient for our construction. The boundary of S intersected with $\bar{A}_+$ has a point y (or many such points) most distant from the $d-1$ dimensional plane perpendicular to ξ and passing through x_0, $R^{d-1}(x_0, \xi)$. Projecting that point (or one of the points) on $R^{d-1}(x_0, \xi)$, we have a point x_1. The $d-2$ dimensional sphere $\{x: |x-x_1| = r\} \cap R^{d-1}(x_0, \xi)$ is now created and on it we can choose d equidistant points which we connect to y chosen above to create our d independent directions ξ_j. The line connecting any point in $A_+ \cap S$ and $R^{d-1}(x_0, \xi)$ in the ξ_j direction will be in S by convexity and will meet $R^{d-1}(x_0, \xi)$ in $\{x: |x-x_0| < 2r\} \cap R^{d-1}(x_0, \xi)$ by the choice of y. If a point in $\{x: |x-x_0| < 2r\}$ is on that line in the ξ_i direction, a segment of length $2r$ in that direction is in S. We choose V_j to be

$$V_j = \{x \in S: \operatorname*{Sup}_{\lambda_i} |\lambda_1 - \lambda_2| > 2r, x + \lambda_1\xi_j, x + \lambda_2\xi_j \in S\}.$$

The above argument now shows that V_j covers $A_+ \cap S$. It is easy to see that (a) and (b) of our choice are trivially satisfied. The cover $V_1, ..., V_j$ will cover $A_+ \cap S$ and hence one can choose a cover that satisfies (c) as well. ∎

5. Remarks

We note that in spite of some similarities, the constructions in Sections 3 and 4 have some differences and the theorems in those sections are only partially interdependent.

In [3, Chap. 12], a comparison between some moduli of smoothness and best polynomial approximation is given. There are the obvious connections between the inequalities investigated here and the relations given there. However, we use here $\tilde{d}(e, x)$ which has a somewhat different definition than $\tilde{d}_S(e, x)$ used in [3]. This should not make any difference as one may observe immediately that

$$\tfrac{1}{2}\tilde{d}_S(e, x) \leqslant \tilde{d}(e, x) \leqslant \tilde{d}_S(e, x) \tag{5.1}$$

and hence, defining $\tilde{\omega}^r_S(f, t)_p$ with $\tilde{d}(e, x)$ replacing $\tilde{d}_S(e, x)$ would lead to an equivalent expression.

In [3, p. 202], the expression $\omega^r_S(f, t)_p$ was also compared to polynomials of best approximation on a simple polytope S. The expression $\omega^r_S(f, t)_p$ is derived by examining only the directions of the edges of the polytope S. While $\omega^r_S(f, t)_p$ and $\tilde{\omega}^r_S(f, t)_p$ are not equivalent for $p = \infty$ (and $p = 1$), they are close enough as [3, Theorem 12.2.3] shows.

If we take the example of the simplex (triangle) $x, y \geqslant 0$, $x + y \leqslant 1$, the direction of the edges is $e_1 = (0, 1)$, $e_2 = (1, 0)$, and $e_3 = (1/\sqrt{2})(1, -1)$. For these edges, one may replace $\tilde{d}_S(e_i, (x, y))$ or $\tilde{d}(e_i, (x, y))$ by $x(1 - x - y)$, $y(1 - x - y)$ and xy for $i = 1, 2$, or 3, respectively. In fact, for $i = 1$ and $i = 2$, the above is $\tilde{d}(e_i, (x, y))$ and for $i = 3$ it is $(1/2)\,\tilde{d}(e_3, (x, y))$. Hence, $\tilde{\omega}^2_S(f, t)$ for the triangle S is obviously equivalent to

$$\sup_{0 < h \leqslant t} \left(\| \Delta^2_{h\sqrt{x(1-x-y)}\,e_1} f \| + \| \Delta^2_{h\sqrt{y(1-x-y)}\,e_2} f \| + \| \Delta^2_{h\sqrt{xy}\,e_3} f \|\right).$$

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